Asymptotic analysis of the Bell polynomials by the ray method

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Abstract

We analyze the Bell polynomials $B_n(x)$ asymptotically as $n \to \infty$. We obtain asymptotic approximations from the differential-difference equation which they satisfy, using a discrete version of the ray method. We give some examples showing the accuracy of our formulas.

Keywords Bell polynomials, asymptotic expansions, Stirling numbers MSC-class: 34E05, 11B73, 34E20

1 Introduction

The Bell polynomials $B_n(x)$ are defined by [1]

$$B_n(x) = \sum_{k=0}^{n} S_k^n x^k, \quad n = 0, 1, \ldots,$$

where S_k^n is a Stirling number of the second kind [2, 24,1,4]. They have the generating function

$$\sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!} = \exp\left[x\left(e^t - 1\right)\right], \tag{1}$$

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from which it follows that

$$B_0(x) = 1 \tag{2}$$

and

$$B_{n+1}(x) = x\left[B_n'(x) + B_n(x)\right], \quad n = 0, 1, \ldots. \tag{3}$$

The asymptotic behavior of $B_n(x)$ was studied by Elbert [3], [4] and Zhao [5], using the saddle point method and (1). In this paper we will use a different approach and analyze (3) instead of (1). The advantage of our method is that no knowledge of a generating function is required and therefore it can be applied to other sequences of polynomials satisfying differential-difference equations [6], [7].

2 Asymptotic analysis

To analyze (3) asymptotically as $n \to \infty$, we use a discrete version of the ray method [8]. Replacing the anszat

$$B_n(x) = \varepsilon^{-n} F(\varepsilon x, \varepsilon n) \tag{4}$$

in (3), we get

$$F(u, v + \varepsilon) = u\left(\varepsilon \frac{\partial F}{\partial x} + F\right), \tag{5}$$

with

$$u = \varepsilon x, \quad v = \varepsilon n \tag{6}$$

and ε is a small parameter. We consider asymptotic solutions for (5) of the form

$$F(u, v) \sim \exp\left[\varepsilon^{-1}\psi(u, v)\right] K(u, v), \tag{7}$$

as $\varepsilon \to 0$. Using (7) in (5) we obtain, to leading order, the eikonal equation

$$e^q - u(p + 1) = 0 \tag{8}$$

and the transport equation

$$\frac{\partial K}{\partial v} + \frac{1}{2}\frac{\partial^2 \psi}{\partial v^2} K - u \exp\left(-\frac{\partial \psi}{\partial v}\right)\frac{\partial K}{\partial u} = 0, \tag{9}$$

where

$$p = \frac{\partial \psi}{\partial x}, \quad q = \frac{\partial \psi}{\partial v}. \tag{10}$$

The initial condition (2), implies

$$\psi(u, 0) = 0, \quad K(u, 0) = 1. \tag{11}$$

To solve (8) we use the method of characteristics, which we briefly review. Given the first order partial differential equation

$$\mathfrak{F}(u, v, \psi, p, q) = 0,$$

with p, q defined in (10), we search for a solution $\psi(u,v)$ by solving the system of "characteristic equations"

$$\begin{aligned} u &= \frac{du}{dt} = \frac{\partial \mathfrak{F}}{\partial p}, \quad v = \frac{dv}{dt} = \frac{\partial \mathfrak{F}}{\partial q}, \\ \dot{p} &= \frac{dp}{dt} = -\frac{\partial \mathfrak{F}}{\partial u} - p\frac{\partial \mathfrak{F}}{\partial \psi}, \quad \dot{q} = \frac{dq}{dt} = -\frac{\partial \mathfrak{F}}{\partial v} - q\frac{\partial \mathfrak{F}}{\partial \psi}, \\ \dot{\psi} &= \frac{d\psi}{dt} = p\frac{\partial \mathfrak{F}}{\partial p} + q\frac{\partial \mathfrak{F}}{\partial q}, \end{aligned}$$

where we now consider $\{u, v, \psi, p, q\}$ to all be functions of the new variables t and s.

For (8), we have

$$\mathfrak{F}(u, v, \psi, p, q) = e^q + p - 2u$$

and therefore the characteristic equations are

$$\dot{u} + u = 0, \quad \dot{v} = e^q, \quad \dot{p} - p = 1, \quad \dot{q} = 0 \tag{12}$$

Solving (12), subject to the initial conditions

$$u(0,s) = s, \quad v(0,s) = 0, \quad p(0,s) = B(s) - 1, \tag{13}$$

we obtain

$$u = se^{-t}, \quad v = Bst, \quad p = Be^t - 1, \quad q = \ln(Bs)$$

where we have used

$$0 = \mathfrak{F}|_{t=0} = e^{q(0,s)} - sB.$$

From (11) and (13) we have

$$\psi(0,s) = 0, \quad K(0,s) = 1, \tag{14}$$

which implies

$$\begin{aligned} 0 = \frac{d}{ds}\psi(0,s) &= p(0,s)\frac{d}{ds}u(0,s) + q(0,s)\frac{d}{ds}v(0,s) \\ &= (B-1)\times 1 + \ln(Bs)\times 0 = B - 1. \end{aligned}$$

Thus,

$$u = se^{-t}, \quad v = st, \quad p = e^t - 1, \quad q = \ln(s). \tag{15}$$

The characteristic equation for ψ is

$$\dot{\psi} = p\dot{u} + q\dot{v} = \left(e^t - 1\right)\left(-se^{-t}\right) + \ln(s)\, s,$$

which together with (14) gives

$$\psi(t,s) = s\left(1 - t - e^{-t}\right) + \ln(s)\, st. \tag{16}$$

We shall now solve the transport equation (9). From (15), we get

$$\frac{\partial t}{\partial u} = -\frac{te^t}{s(t+1)}, \quad \frac{\partial t}{\partial v} = \frac{1}{s(t+1)}, \quad \frac{\partial s}{\partial u} = \frac{e^t}{t+1}, \quad \frac{\partial s}{\partial v} = \frac{1}{t+1} \tag{17}$$

and therefore,

$$\frac{\partial^2\psi}{\partial v^2} = \frac{\partial q}{\partial v} = \frac{\partial q}{\partial t}\frac{\partial t}{\partial v} + \frac{\partial q}{\partial s}\frac{\partial s}{\partial v} = \frac{1}{s\,(t+1)}. \tag{18}$$

Using (17)-(18) to rewrite (9) in terms of t and s, we have

$$\dot{K} + \frac{1}{2\,(t+1)}K = 0$$

with solution

$$K(t,s) = \frac{1}{\sqrt{t+1}}, \tag{19}$$

where we have used (14).

Solving for t, s in (15), we obtain

$$t = \mathrm{LW}\left(\frac{v}{u}\right), \quad s = \frac{v}{\mathrm{LW}\left(\frac{v}{u}\right)}, \tag{20}$$

where $\mathrm{LW}(\cdot)$ denotes the Lambert-W function [9], defined by

$$\mathrm{LW}(z)\exp\left[\mathrm{LW}(z)\right] = z.$$

Replacing (20) in (16) and (19), we get

$$\psi(u,v) = \frac{v}{\mathrm{LW}\left(\frac{v}{u}\right)} + v\ln\left[\frac{v}{\mathrm{LW}\left(\frac{v}{u}\right)}\right] - (u+v),$$

$$K(u,v) = \frac{1}{\sqrt{\mathrm{LW}\left(\frac{v}{u}\right)+1}}$$

and from (7) we find that

$$F(u,v) \sim \exp\left\{\frac{v/\varepsilon}{\mathrm{LW}\left(\frac{v}{u}\right)} + \frac{v}{\varepsilon}\ln\left[\frac{v}{\mathrm{LW}\left(\frac{v}{u}\right)}\right] - (\frac{u+v}{\varepsilon})\right\}\frac{1}{\sqrt{\mathrm{LW}\left(\frac{v}{u}\right)+1}}, \tag{21}$$

as $\varepsilon \to 0$. Using (6) and (21) in (4), we conclude that

$$B_n(x) \sim \exp\left\{\frac{n}{\mathrm{LW}\left(\frac{n}{x}\right)} + n\ln\left[\frac{n}{\mathrm{LW}\left(\frac{n}{x}\right)}\right] - (x+n)\right\}\frac{1}{\sqrt{\mathrm{LW}\left(\frac{n}{x}\right)+1}}, \tag{22}$$

as $n \to \infty$.

Remark 1 *The function* $\mathrm{LW}(z)$ *has two real-valued branches for* $-e^{-1} \leq z < 0$, *denoted by* $\mathrm{LW}_0(z)$ *(the principal branch of* LW*) and* $\mathrm{LW}_{-1}(z)$, *satisfying*

$$\mathrm{LW}_0 : \left[-e^{-1}, 0\right) \to [-1, 0), \quad \mathrm{LW}_{-1} : \left[-e^{-1}, 0\right) \to (-\infty, -1],$$

with

$$\mathrm{LW}_0\left(-e^{-1}\right) = -1 = \mathrm{LW}_{-1}\left(-e^{-1}\right).$$

For $z \geq 0$, $\mathrm{LW}(z)$ *has only one real-valued branch*

$$\mathrm{LW}_0 : [0,\infty) \to [0,\infty)$$

and for $z < -e^{-1}$, $\mathrm{LW}_0(z)$ *and* $\mathrm{LW}_{-1}(z)$ *are complex conjugates. Therefore, for (22) to be well defined, we need to consider three separate regions:*

1. *An exponential region for* $x > 0$ *or* $x < -en$. *Here we have*

$$B_n(x) \sim \Phi_n(x;0), \quad n \to \infty, \tag{23}$$

where

$$\Phi_n(x;k) = \exp\left\{\frac{n}{\mathrm{LW}_k\left(\frac{n}{x}\right)} + n\ln\left[\frac{n}{\mathrm{LW}_k\left(\frac{n}{x}\right)}\right] - (x+n)\right\}\frac{1}{\sqrt{\mathrm{LW}_k\left(\frac{n}{x}\right)+1}}.$$

2. *An oscillatory region for* $-en < x < 0$. *In this interval,*

$$B_n(x) \sim \Phi_n(x;0) + \Phi_n(x;-1), \quad n \to \infty. \tag{24}$$

In Figure 1 (a) we plot $B_5(x)$ *and the asymptotic approximations (23) (+++) and (24) (ooo), all multiplied by* $e^{-|x|}$ *for scaling purposes, in the interval* $(-10,10)$. *We see that our formulas are quite accurate even for small values of* n *and that the transition between (23) and (24) is smooth.*

3. *A transition region for* $x \simeq -en$. *We will analyze this region in the next section.*

In Figure 1 (b) we plot $B_5(x)$ and (23) (+++) and (24) (ooo), all multiplied by e^x, in the interval $(-20,0)$. We observe that the approximations (23) and (24) break down in the neighborhood of $-e5 \simeq -13,59$.

2.1 The transition region

When $x = -en$, the quantity $\mathrm{LW}\left(\frac{n}{x}\right)+1$ vanishes and (23) is no longer valid. To find an asymptotic approximation in a neighborhood of $-en$, we introduce the stretched variable β defined by

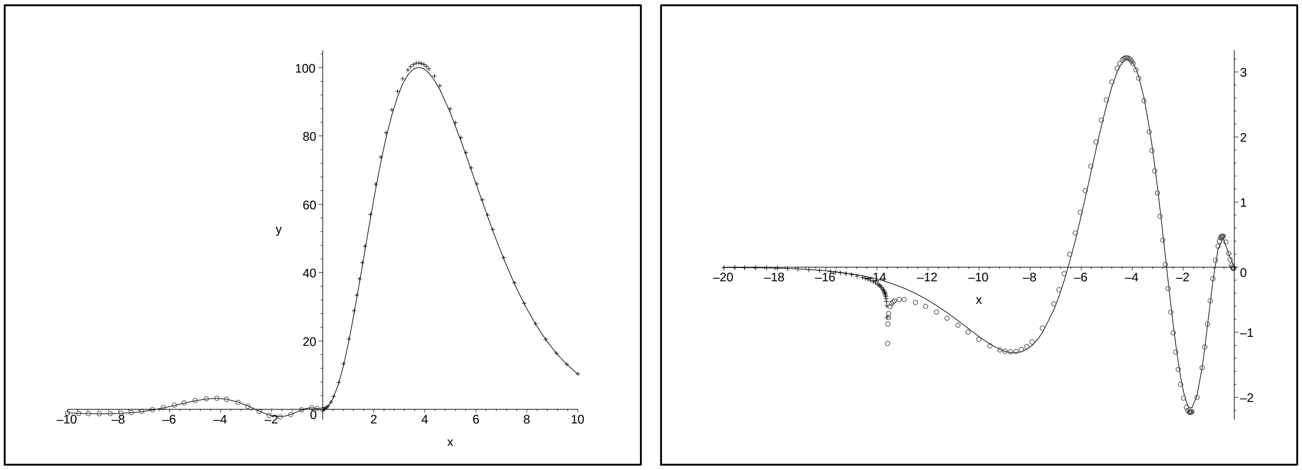
$$x = -en - \beta n^{\frac{1}{3}}, \quad \beta = O(1). \tag{25}$$

For values of z close to $z_0 = -e^{-1}$, the Lambert-W function can be approximated by [9, (4.22)]

$$\mathrm{LW}(z) \sim -1 + \sqrt{2e(z-z_0)} - \frac{2}{3}e(z-z_0) + \frac{11}{36}\sqrt{2e^3(z-z_0)^3}, \quad z \to -e^{-1}. \tag{26}$$

Using(25) in (26), we have,

$$\mathrm{LW}\left(\frac{n}{-en-\beta n^{\frac{1}{3}}}\right) \sim -1 + \sqrt{2e^{-1}\beta}n^{-\frac{1}{3}} - \frac{2}{3}e^{-1}\beta n^{-\frac{2}{3}} - \frac{7}{36}\sqrt{2e^{-3}\beta^3}n^{-1}, \quad \beta \to 0. \tag{27}$$



(a) (b)

Figure 1: A comparison of the exact (solid curve) and asymptotic (ooo), (+++) values of $B_5(x)$.

Hence,

$$\exp\left\{\frac{n}{\mathrm{LW}_k\left(\frac{n}{x}\right)}+n\ln\left[\frac{n}{\mathrm{LW}_k\left(\frac{n}{x}\right)}\right]-(x+n)\right\}\sim\varphi(\beta,n)\,,\quad \beta\to 0,$$

for $k=0,1$ with $x=-en-\beta n^{\frac{1}{3}}$ and

$$\varphi(\beta,n)=(-1)^n\exp\left\{\left[\ln(n)+e-2\right]n-\left(e^{-1}-1\right)\beta n^{\frac{1}{3}}\right\}. \tag{28}$$

We now consider solutions for (3) of the form

$$B_n(x)=\varphi(\beta,n)\,\Lambda(\beta)=\varphi\left[-\left(e+\frac{x}{n}\right)n^{\frac{2}{3}},n\right]\Lambda\left[-\left(e+\frac{x}{n}\right)n^{\frac{2}{3}}\right], \tag{29}$$

for some function $\Lambda(\beta)$. Replacing (29) in (3) and using (25) we obtain, to leading order

$$\Lambda''-2e^{-3}\beta\Lambda=0,$$

with solution

$$\Lambda(\beta)=C_1\mathrm{Ai}\left(2^{\frac{1}{3}}e^{-1}\beta\right)+C_2\mathrm{Bi}\left(2^{\frac{1}{3}}e^{-1}\beta\right), \tag{30}$$

where $\mathrm{Ai}(\cdot)$, $\mathrm{Bi}(\cdot)$ are the Airy functions.

To determine the constants C_1, C_2 in (30), we shall match (23) with (29). Using (25) and (27) in (23), we have

$$B_n(x)\sim\varphi(\beta,n)\exp\left(-\frac{2}{3}\sqrt{2}e^{-\frac{3}{2}}\beta^{\frac{3}{2}}\right)\left(2e^{-1}\beta\right)^{\frac{1}{4}}n^{-\frac{1}{6}},\quad \beta\to 0^+. \tag{31}$$

On the other hand, the Airy functions have the well known asymptotic approximations [2, (10.4.59, 10.4.63)]

$$\mathrm{Ai}\,(z) \sim \frac{1}{2\sqrt{\pi}} \exp\left(-\frac{2}{3} z^{\frac{3}{2}}\right) z^{-\frac{1}{4}}, \quad z \to \infty,$$
$$\mathrm{Bi}\,(z) \sim \frac{1}{\sqrt{\pi}} \exp\left(\frac{2}{3} z^{\frac{3}{2}}\right) z^{-\frac{1}{4}}, \quad z \to \infty$$

and therefore we conclude that

$$C_1 = \sqrt{\pi} 2^{\frac{5}{6}} n^{\frac{1}{6}}, \quad C_2 = 0. \tag{32}$$

Replacing (30) and (32) in (29), we find that for $x \simeq -en$, we have

$$B_n\,(x) \sim \sqrt{\pi} 2^{\frac{5}{6}} n^{\frac{1}{6}} \varphi\,(\beta, n)\,\mathrm{Ai}\left(2^{\frac{1}{3}} e^{-1} \beta\right), \qquad n \to \infty.$$

This concludes the asymptotic analysis of $B_n(x)$ for large n.

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References

[1] E. T. Bell, Exponential polynomials, Ann. of Math. (2) 35 (2) (1934) 258–277.

[2] M. Abramowitz, I. A. Stegun (Eds.), Handbook of mathematical functions with formulas, graphs, and mathematical tables, Dover Publications Inc., New York, 1992.

[3] C. Elbert, Weak asymptotics for the generating polynomials of the Stirling numbers of the second kind, J. Approx. Theory 109 (2) (2001) 218–228.

[4] C. Elbert, Strong asymptotics of the generating polynomials of the Stirling numbers of the second kind, J. Approx. Theory 109 (2) (2001) 198–217.

[5] Y.-Q. Zhao, A uniform asymptotic expansion of the single variable Bell polynomials, J. Comput. Appl. Math. 150 (2) (2003) 329–355.

[6] D. E. Dominici, Asymptotic analysis of the Hermite polynomials from their differential-difference equation (2007) To appear in the Journal of Difference Equations and Applications.

[7] D. E. Dominici, Asymptotic analysis of the asymptotic analysis of generalized Hermite polynomials (2007) Submitted.

[8] E. Giladi, J. B. Keller, Eulerian number asymptotics, Proc. Roy. Soc. London Ser. A 445 (1924) (1994) 291–303.

[9] R. M. Corless, G. H. Gonnet, D. E. G. Hare, D. J. Jeffrey, D. E. Knuth, On the Lambert W function, Adv. Comput. Math. 5 (4) (1996) 329–359.